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DERIVATIONS HAVING DIVERGENCE ZERO ON $R[X,Y]$

BY

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ABSTRACT

In this paper it is proved that for any $\mathbb{Q}$-algebra R any locally nilpotent R-derivation D on $R[X,Y]$ having divergence zero and $1 \in (D(X), D(Y))$ (i) has a slice, and (ii) $A^D = R[P]$ for some P. Furthermore, it is shown that any surjective R-derivation on $R[X,Y]$ having divergence zero is locally nilpotent. Connections with the Jacobian Conjecture are made.

1. Introduction

Locally nilpotent R-derivations on the polynomial ring $R[X,Y]$ where R is a UFD containing $\mathbb{Q}$ were studied by Daigle and Freudenburg in [3]. The more general situation where R is a (normal) Noetherian domain containing $\mathbb{Q}$ was studied by Bhatwadekar and Dutta in [2]. They showed, amongst other things, that if D is a locally nilpotent derivation on $R[X,Y]$ such that the ideal generated by $D(X)$ and $D(Y)$ contains 1, then $R[X,Y]^D$ is a polynomial ring in one variable over R and $R[X,Y]$ is a polynomial ring in one variable over $R[X,Y]^D$. In particular, this implies that D has a slice in $R[X,Y]$.

In this paper we generalise this result to arbitrary $\mathbb{Q}$-algebras R in the sense that we consider locally nilpotent derivations having divergence zero (in the domain case any locally nilpotent derivation has divergence zero).

Also, we generalise a result of Stein in [8], asserting that any surjective k-derivation on $k[X,Y]$ (k a field of characteristic zero) is locally nilpotent, to surjective divergence zero R-derivations on $R[X,Y]$ where R is an arbitrary $\mathbb{Q}$-algebra.

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At the end of this paper we relate this result to the Jacobian Conjecture. In fact the importance of divergence zero derivations for this conjecture will be described in a forthcoming paper of the second author.

2. Preliminaries

In the rest of this paper R denotes a commutative $\mathbb{Q}$-algebra, A an R-algebra containing R and D an R-derivation on A. The set of elements $a \in A$ satisfying $Da = 0$ is denoted by A^D. It is an R-subalgebra of A called the **ring of constants** of D. An element $s \in A$ satisfying $Ds = 1$ is called a **slice** of D and finally D is called **locally nilpotent** if for every $a \in A$ there exists $n \in \mathbb{N}$ with $D^n a = 0$.

2.1 Derivations having a slice.

PROPOSITION 2.1: *Let D be a locally nilpotent R-derivation on A having a slice $s \in A$. Then $A = A^D[s]$, a polynomial ring in s over A^D and $D = d/ds$ on A.*

For a proof we refer to [6], [9] or [4], proposition 1.3.21.

COROLLARY 2.2: *Let D be a locally nilpotent R-derivation on A. Then D has a slice in A if and only if D is surjective.*

Proof: Follows immediately from Proposition 2.1 since d/ds on $A^D[s]$ is surjective. ∎

To formulate the next lemmas we introduce some notation. Let I be an ideal of R. The element $a + AI$ in A/AI will be denoted by a_I and the induced derivation on A/AI by D_I.

LEMMA 2.3: *Let D be an R-derivation on A. Let $I, J \subset R$ be ideals of R and suppose D_I has a slice and D_J is surjective. Then D_{IJ} has a slice.*

Proof: There exists $s \in A$ such that $D_I(s_I) = 1$ and hence $D(s) = 1 + f$ for some $f \in IA$. Write $f = \sum f_\alpha a_\alpha$, where $f_\alpha \in I$ and $a_\alpha \in A$. Since D_J is surjective there exists $F_\alpha \in A$ such that $D(F_\alpha) = a_\alpha + h_\alpha$, where $h_\alpha \in JA$. Denote $S := s - \sum f_\alpha F_\alpha$. Then

$$\begin{aligned} D(S) &= D(s - \sum f_\alpha F_\alpha) \\ &= D(s) - \sum f_\alpha D(F_\alpha) \\ &= 1 + f - \sum (f_\alpha a_\alpha + f_\alpha h_\alpha) \\ &= 1 - \sum f_\alpha h_\alpha, \end{aligned}$$

and since $f_\alpha h_\alpha \in IJA$ we have $D_{IJ}(S_{IJ}) = 1$. ∎

LEMMA 2.4: *Let D_{I_i} be surjective for the ideals $I_1, \ldots, I_r \subset R$. Then $D_{I_1 \cdot \ldots \cdot I_r}$ is also surjective.*

Proof: It is enough to show that if D_I, D_J are surjective then D_{IJ} is too. Let $a \in A$ be arbitrary. There exists $b \in A$ such that $D_I(b_I) = a_I$, hence $D(b) = a + i$ where $i \in IA$. Write $i = \sum_{k=0}^t i_k c_k$ where $i_k \in I$, $c_k \in A$. Then for every c_k there exists for some d_k such that $D(d_k) = c_k + j_k$ for some $j_k \in JA$ since D_J surjective. Now $D(b - \sum_{k=0}^t i_k d_k) = a - \sum_{k=0}^t i_k j_k$. Since $\sum_{k=0}^t i_k j_k \in IJA$ we're done. ∎

LEMMA 2.5: *Let D be a locally nilpotent R-derivation on A. If $I_1, \ldots, I_r \subset R$ are ideals of R and D_{I_i} has a slice for all i, then $D_{I_1 \cdot \ldots \cdot I_r}$ has a slice too.*

Proof: It is enough to show that if D_I, D_J both have a slice then D_{IJ} has one too. By Corollary 2.2, D_I and D_J are surjective. By Lemma 2.4, D_{IJ} is surjective. In particular, D_{IJ} has a slice. ∎

LEMMA 2.6: *If $I_1, \ldots, I_r \subset R$ are ideals of R and D_{I_i} is locally nilpotent for all i, then $D_{I_1 \cdot \ldots \cdot I_r}$ is locally nilpotent too.*

Proof: It is enough to show that if D_I, D_J are locally nilpotent then D_{IJ} is locally nilpotent. Let $a \in A$. One knows there exists $N \in \mathbb{N}$ such that $D_I^N(a_I) = 0$, hence $D^N(a) = \sum_{k=0}^t i_k b_k$ where $i_k \in I, b_k \in A$. Now there exists $M_k \in \mathbb{N}$ such that $D^{M_k}(b_k) \in JA$. Let $M = \max_k(M_k)$. Then $D^{N+M}(a) = D^M(\sum_{k=0}^t i_k b_k) = \sum_{k=0}^t i_k D^M(b_k) \in IJA$. ∎

2.2 POLYNOMIAL AUTOMORPHISMS OVER A COMMUTATIVE RING. Let $n \geq 0$. Then $R^{[n]}$ denotes the polynomial ring $R[X] := R[X_1, \ldots, X_n]$. An R-homomorphism of $R^{[n]}$ is completely determined by the images of the X_i. So we get a one-to-one correspondence between the R-homomorphisms of $R^{[n]}$ and the n-tuples $F = (F_1, \ldots, F_n) \in (R^{[n]})^n$. Such an n-tuple we call a **polynomial map over** R. Restricting the above correspondence to the R-automorphisms of $R^{[n]}$ we get a one-to-one correspondence with the so-called *invertible* (over R) *polynomial maps*. It is well-known that F is invertible over R if and only if $R[X] = R[F_1, \ldots, F_n]$.

Now let $F = (F_1, \ldots, F_n) \in (R^{[n]})^n$ and $\mathfrak{p}$ be a prime ideal of R. Reducing all coefficients of all F_i modulo $\mathfrak{p}$ we get a polynomial map over $R/\mathfrak{p}$, which we denote by $F_\mathfrak{p}$. So $F_\mathfrak{p} = (F_{1\mathfrak{p}}, \ldots, F_{n\mathfrak{p}})$. Obviously, if F is invertible over R, hence so is $F_\mathfrak{p}$ over $R/\mathfrak{p}$. In section 3 below we need the following converse.

PROPOSITION 2.7: *Let R be noetherian and $F \in (R^{[n]})^n$. If $F_\mathfrak{p}$ is invertible over $R/\mathfrak{p}$ for all minimal prime ideals of the nilradical η, then F is invertible over R.*

Proof: Since R is noetherian, $\eta = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$, a finite intersection of all minimal prime ideals of R. Furthermore, $\eta^e = 0$ for some $e \geq 1$. If $\mathfrak{p}$ is one of the $\mathfrak{p}_i$, then the hypothesis on $F_\mathfrak{p}$ implies that

$$R[X] \subset \mathfrak{p}R[X] + R[F], \quad \text{where } R[F] = R[F_1, \ldots, F_n].$$

So

$$R[X] \subset \mathfrak{p}_1 R[X] + R[F] \subset \mathfrak{p}_1(\mathfrak{p}_2 R[X] + R[F]) + R[F] \subset \mathfrak{p}_1\mathfrak{p}_2 R[X] + R[F].$$

Continuing in this way we get

$$R[X] \subset \mathfrak{p}_1 \cdots \mathfrak{p}_r R[X] + R[F] \subset \eta R[X] + R[F]. \tag{1}$$

Applying (1) again we get

$$R[X] \subset \eta(\eta R[X] + R[F]) + R[F] \subset \eta^2 R[X] + R[F].$$

Continuing in this way and using that $\eta^e = 0$ we get

$$R[X] \subset \eta^e R[X] + R[F] = R[F].$$

So $R[X] = R[F]$, i.e., F is invertible over R. ∎

To conclude this section of preliminaries we recall a well-known result concerning locally nilpotent derivations on $R^{[n]}$ in case R is a domain. Let D be an R-derivation on $R^{[n]} = R[X_1, \ldots, X_n]$. Then D is of the form $a_1\partial_1 + \cdots + a_n\partial_n$ with $a_i \in R^{[n]}$ for all i. The **divergence** of D, denoted div D, is defined as the element $\sum_{i=1}^n \partial_i(a_i)$ in $R^{[n]}$.

PROPOSITION 2.8: *If R is a domain and D a locally nilpotent R-derivation on $R^{[n]}$, then* $\operatorname{div} D = 0$.

Since the authors do not know of any reference except proposition 1.3.51 in [4], we include a short proof.

Proof: Introduce a new variable T and consider $R^{[n+1]} = R[X, T]$. Extend D to an R-derivation $\widetilde{D}$ on $R^{[n+1]}$ by putting $\widetilde{D}(T) = 0$. Obviously $\widetilde{D}$, and hence also

$T\widetilde{D}$, is locally nilpotent on $R^{[n+1]}$. Consequently $F := \exp T\widetilde{D} \in \mathrm{Aut}_R\, R^{[n+1]}$. Since $F_i = \exp TD(X_i) = X_i + D(X_i)T + \cdots$ for all $1 \le i \le n$ and $F_{n+1} = T$, one easily verifies that

$$J_{X_1,\dots,X_n,T}F = I_{n+1} + \begin{pmatrix} \left(\frac{\partial D(X_i)}{\partial X_j}\right)_{1\le i,j\le n} & * \\ 0 & 0 \end{pmatrix} T + \cdots,$$

which implies that the coefficient of T in the T-expansion of $j(F) := \det J_{X_1,\dots,X_n,T}F$ equals

$$\sum \frac{\partial D(X_i)}{\partial X_i} = \operatorname{div} D.$$

On the other hand, since $F \in \mathrm{Aut}_R\, R^{[n+1]}$ it follows that $j(F) \in (R^{[n+1]})^* = R^*$ (since R is a domain!). In particular, the T-coefficient of $j(F)$ equals zero. So $\operatorname{div} D = 0$, as desired. ∎

3. Divergence zero derivations

Throughout this section let $A = R[X,Y]$ and D be a non-zero R-derivation on A with divergence zero. Then it is well-known that $D = P_Y\partial_X - P_X\partial_Y$ for some $P \in A$ (where $P_X = \partial_X(P)$ and $P_Y = \partial_Y(P)$ are the derivatives of P), which is unique if one assumes $P(0,0) = 0$. We denote this element by $P(D)$. We say that R has property $B(R)$ if and only if the following holds:

$B(R)$ Any locally nilpotent derivation D on A with $\operatorname{div}(D) = 0$ and $1 \in (D(X), D(Y))$ has a slice and satisfies $A^D = R[P(D)]$.

The main aim of this section is to show that $B(R)$ holds for any $\mathbb{Q}$-algebra R (Theorem 3.5). We first reduce to the case that R is Noetherian. Therefore, let R' be the $\mathbb{Q}$-subalgebra of R generated by the coefficients of the polynomials P, a and b where a, b are such that $1 = aP_X + bP_Y$. Notice that R' is noetherian, regardless of R. Write $A' = R'[X,Y]$, D' the restriction of D to A'.

LEMMA 3.1: *If D' has a slice and ${A'}^{D'} = R'[P]$, then D has a slice and $A^D = R[P]$.*

Proof: Let $S \in A'$ such that $D'(S) = 1$. Then since $A' \subseteq A$ we have $S \in A$ and $D(S) = D'(S) = 1$. So let ${A'}^{D'} = R'[P]$. By Proposition 2.1 we get $R'[X,Y] = A' = {A'}^{D'}[S] = R'[P,S]$. So there exist $F, G \in R'[X,Y]$ such that $F(P,S) = X$ and $G(P,S) = Y$. But since all is contained in $R[X,Y]$ we have

$$R[X,Y] = R[F(P,S), G(P,S)] \subseteq R[P,S] \subseteq R[X,Y].$$

Hence $A^D = R[P,S]^D = R[P]$. ∎

To prove $B(R)$ for Noetherian domains containing $\mathbb{Q}$, we first need a lemma from [3]

LEMMA 3.2: *Let R be a domain containing $\mathbb{Q}$ and $P \in R[X,Y]$ such that $1 \in (P_X, P_Y)$. Then $K[P] \cap R[X,Y] = R[P]$, where $K = Q(R)$, its field of fractions.*

Proof: If $K[P] \cap R[X,Y] \not\subseteq R[P]$, then there exists an $F \in K[T] \setminus R[T]$ with $F(P) \in R[X,Y]$. Choose one of minimal degree. Observe that $F(P) \in R[X,Y]$ implies that $F'(P)P_X$ and $F'(P)P_Y$ belong to $R[X,Y]$.

Since there are $g, h \in R[X,Y]$ with $P_X g + P_Y h = 1$, we deduce $F'(P) = F'(P)P_X g + F'(P)P_Y h \in R[X,Y]$. So $F'(T) \in K[T]$ and $F'(P) \in R[X,Y]$, thus by minimality of the degree of F we must conclude that $F' \in R[T]$. Now write $F = \sum_{i=0}^{d} f_i T^i$; then $F' \in R[T]$ implies (since R is a $\mathbb{Q}$-algebra) that $f_i \in R$ for all $i \geq 1$, thus yielding $f_0 = F(P) - \sum_{i=1}^{d} f_i P^i \in R[X,Y] \cap K = R$, contradicting the assumption that $F \notin R[T]$. ∎

Now we can prove the next theorem:

LEMMA 3.3: *Let R be a Noetherian domain containing $\mathbb{Q}$ and D a locally nilpotent derivation on $R[X,Y]$ with $1 \in (D(X), D(Y))$. Then $R[X,Y]^D = R[P]$ for some $P \in R[X,Y]$ and D has a slice $t \in R[X,Y]$.*

Proof: Let K be the quotient field of R. Extend D to $K[X,Y]$ the natural way. We know by Rentschler's theorem (see for example [7] and [4], Th. 1.2.25) that there is a $Q \in K[X,Y]$ with $K[X,Y]^D = K[Q]$. Because D is locally nilpotent, we know by Proposition 2.8 that $\mathrm{div}(D) = 0$, so there is a $P \in R[X,Y]$ with $D(X) = P_Y$ and $D(Y) = -P_X$. This means that $D(P) = 0$ and, as a consequence, $P \in K[X,Y]^D = K[Q]$. So write $P = g(Q)$ with $g \in K[T]$. We now have $P_X = g'(Q)Q_X$ and $P_Y = g'(Q)Q_Y$. Notice that $(P_Y, P_X) = (D(X), D(Y)) = (1)$ (also in $K[X,Y]$), which means that $g'(Q) \in K^*$. Then there are $\lambda, \mu \in K$, $\lambda \neq 0$ satisfying $P = g(Q) = \lambda Q + \mu$, yielding $K[P] = K[Q]$. By the previous lemma, $R[X,Y]^D = K[X,Y]^D \cap R[X,Y] = K[P] \cap R[X,Y] = R[P]$. Hence we proved our first claim. Now we can use Theorem 4.7 in [2] to conclude that

$$R[X,Y] = R[P][s] \quad \text{for some } s \in R[X,Y]. \tag{1}$$

This means that $f\colon R[X,Y] \longrightarrow R[X,Y]$ defined by $f(X) = P(X,Y)$ and $f(Y) = s(X,Y)$ satisfies $f \in \mathrm{Aut}_R\, R[X,Y]$. A well-known consequence is that

$$\det Jf \in R[X,Y]^* = R^*. \tag{2}$$

But this determinant is equal to $-P_Y s_X + P_X s_Y = -D(s)$. So $D(s) \in R^*$, whence $t := s/D(s)$ satisfies $D(t) = 1$ and we are done. ∎

Combining Lemma 3.1 and Theorem 3.3 we have

THEOREM 3.4: *Let R be any domain containing $\mathbb{Q}$. Then $B(R)$ holds.*

Now we are able to prove the main theorem of this section.

THEOREM 3.5: *Let R be any $\mathbb{Q}$-algebra. Then $B(R)$ holds.*

Proof: Let $D = P_Y\partial_X - P_X\partial_Y$ be an arbitrary locally nilpotent derivation on $R[X,Y]$ with div $D = 0$ and $1 \in (P_X, P_Y)$. We have to prove that D has a slice and that $A^D = R[P]$. By Lemma 3.1 we may assume that R is Noetherian. So the nilradical η of R can be written as $\eta = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$, where the $\mathfrak{p}_i$ run through all minimal prime ideals of R and $\eta^e = 0$ for some $e \geq 1$.

(i) First we show that D has a slice in A. Therefore observe that by Theorem 3.4, $D_{\mathfrak{p}_i}$ has a slice in $R/\mathfrak{p}_i[X,Y]$ for all i. So by Lemma 2.5, D_η has a slice in $R/\eta[X,Y]$. Then again by Lemma 2.5, D_{η^e} has a slice in $R/\eta^e[X,Y]$. Since $\eta^e = 0$, this means that D has a slice, say s in $R[X,Y] = A$.

(ii) Finally we claim: $R[P,s] = R[X,Y]$, which upon using $Ds = 1$ and $DP = 0$ implies that $R[X,Y]^D = R[P]$ as desired. To see the claim it suffices by Proposition 2.7 to see that each $F_{\mathfrak{p}_i}$ is invertible over $R/\mathfrak{p}_i$, where $F = (P,s)$. However, by Theorem 3.4 we know that $R/\mathfrak{p}_i[X,Y]^{D_{\mathfrak{p}_i}} = R/\mathfrak{p}_i[P_{\mathfrak{p}_i}]$ and obviously, $D_{\mathfrak{p}_i}(s_{\mathfrak{p}_i}) = 1$. So by Proposition 2.1 we get $R/\mathfrak{p}_i[X,Y] = R/\mathfrak{p}_i[P_{\mathfrak{p}_i}, s_{\mathfrak{p}_i}]$, i.e., $F_{\mathfrak{p}_i}$ is invertible over $R/\mathfrak{p}_i$. ∎

4. Surjective derivations and the two-dimensional Jacobian Conjecture

In this section we consider surjective R-derivations on $R[X_1, X_2]$ having divergence zero. The main result is

THEOREM 4.1: *Let R be any $\mathbb{Q}$-algebra. Then any surjective R-derivation on $R[X_1, X_2]$ having divergence zero is locally nilpotent.*

To prove this theorem we recall a result of [4]. Let D be a non-zero derivation on $R[X_1, X_2]$. Put $d := \max_{i,j} \deg_{X_i} D(X_j)$.

PROPOSITION 4.2 ([4], Theorem 1.3.52): *Let R be a domain containing $\mathbb{Q}$. Then D is locally nilpotent if and only if $D^{d+2}(X_i) = 0$ for $i = 1, 2$.*

One can easily deduce Proposition 4.2 by reducing it first to the case that R is a field, and then use P from the proof of Theorem 3.3 and apply Corollary 1.5 from [5] to show that $D^{\deg_Y(P)+1}(X) = 0$.

Proof of Theorem 4.1: (i) If R is a field, the result was proved by Stein in [8]. From this one easily deduces the case when R is a domain.

(ii) Now let D be a surjective R-derivation on $R[X_1, X_2]$ with divergence zero and $\mathfrak{p}$ a prime ideal in R. Then $D_{\mathfrak{p}}$ is a surjective $R/\mathfrak{p}$-derivation on $R/\mathfrak{p}[X_1, X_2]$ having divergence zero. So by (i) and Proposition 4.2 it follows that $D^{d+2}(X_i) \in \mathfrak{p}R[X_1, X_2]$ for $i = 1, 2$ (where d is as defined above before Proposition 4.2). Since this holds for all $\mathfrak{p}$ we get that $D^{d+2}(X_i) \in \eta R[X_1, X_2]$ for $i = 1, 2$. So D_η is locally nilpotent on $R/\eta[X_1, X_2]$. Then the result follows from Lemma 4.3. ∎

LEMMA 4.3: *If D is an R-derivation on $R[X] := R[X_1, \ldots, X_n]$ such that D_η is a locally nilpotent R/η-derivation on $R/\eta[X]$, then D is locally nilpotent.*

Proof: (i) Let R_0 be the subring of R generated by all coefficients appearing in the polynomials $D(X_1), \ldots, D(X_n)$. Then R_0 is noetherian and the restriction of D to $R_0[X]$, which we denote by D_0, is an R_0-derivation of $R_0[X]$. Observe that D is locally nilpotent if and only if D_0 is locally nilpotent. Furthermore, the nilradical of R_0 equals $R_0 \cap \eta$. Therefore, we can replace R by R_0 and we may assume without loss of generality that R is noetherian.

(ii) So assume that R is noetherian. Then there exists $e \geq 1$ such that $\eta^e = 0$. Since D_η is locally nilpotent, it follows from Lemma 2.6 that D $(= D_{\eta^e})$ is locally nilpotent too. ∎

To conclude this paper we wish to relate Theorem 4.1 to the two-dimensional Jacobian Conjecture. Recall that one can generalize the usual n-dimensional complex Jacobian Conjecture to $JC(R, n)$: If $F \in (R^{[n]})^n$ with $\det JF \in (R^{[n]})^*$, then $R[X] = R[F]$.

It was shown in [1] (see also [4]) that for each $n \geq 1$, $JC(\mathbb{C}, n)$ implies $JC(R, n)$ for any $\mathbb{Q}$-algebra R. In particular, $JC(\mathbb{C}, 2)$ implies $JC(R, 2)$. This enables us to prove

PROPOSITION 4.4: *There is equivalence between*

(i) $JC(\mathbb{C}, 2)$ *is true.*

(ii) *For every $\mathbb{Q}$-algebra R, every R-derivation D on $R[X, Y]$ with $\operatorname{div} D = 0$ and $1 \in \operatorname{Im} D$ is locally nilpotent.*

(iii) *Every $\mathbb{C}$-derivation D on $\mathbb{C}[X, Y]$ with $\operatorname{div} D = 0$ and $1 \in \operatorname{Im} D$ is locally nilpotent.*

Proof: (i) $\to$ (ii) Let D be an R-derivation with div $D = 0$. So $D = P_Y \partial_X - P_X \partial_Y$ for some $P \in R[X, Y]$. Since $1 \in \operatorname{Im} D$ there exists $s \in R[X, Y]$ with $Ds =$

1, i.e., $\det J(s,P) = 1$. Since as observed above $JC(\mathbb{C},2)$ implies $JC(R,2)$, we deduce that $R[s,P] = R[X,Y]$. Consequently, D is locally nilpotent on $R[X,Y]$.

(ii) $\rightarrow$ (iii) is obvious. Finally, assume (iii) and let $F = (F_1, F_2) \in \mathbb{C}[X,Y]^2$ with $\det JF = 1$. Then $\partial/\partial F_1$ $(= F_{2Y}\partial_X - F_{2X}\partial_Y)$ has divergence zero and

$$1 = \frac{\partial}{\partial F_1}(F_1) \in \operatorname{Im} \frac{\partial}{\partial F_1}.$$

So by hypothesis $\partial/\partial F_1$ is locally nilpotent. Similarly, $\partial/\partial F_2$ is locally nilpotent. Then it is well-known (see [1] or [4], proposition 2.2.10) that F is invertible over $\mathbb{C}$. So $JC(\mathbb{C},2)$ holds. ∎

QUESTION 1: *Can one give a finite number of elements $a_1, \ldots, a_m$ in $R[X,Y]$ such that $a_i \in \operatorname{Im}(D)$ for all i implies that D is surjective (of course assuming $\operatorname{div}(D) = 0$)?*

Or more concretely:

QUESTION 2: *Does $\{1, X, Y\} \subset \operatorname{Im}(D)$ imply that D is surjective?*

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